L’Hôpital’s Rule & Indeterminate Forms

In calculus and other branches of mathematical analysis, limits involving an algebraic combination of functions in an independent variable may often be evaluated by replacing these functions by their limits; if the expression obtained after this substitution does not provide sufficient information to determine the original limit, then it is said to assume an indeterminate form. More specifically, an indeterminate form is a mathematical expression involving 0, $1\,\infty$, obtained by applying the algebraic limit theorem in the process of attempting to determine a limit, which fails to restrict that limit to one specific value or infinity (if a limit is confirmed as infinity, then it is not indeterminate since the limit is determined as infinity) and thus does not yet determine the limit being sought

There are seven indeterminate forms which are typically considered in the literature

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \times \infty$
- $\infty \times \infty$
- 0^0
- 1^∞
- ∞^0

1 L’Hôpital’s rule

In mathematics, more specifically calculus, L’Hôpital’s rule or L’Hospital’s rule provides a technique to evaluate limits of indeterminate forms. Application (or repeated application) of the rule often converts an indeterminate form to an expression that can be easily evaluated by substitution. The rule is named after the 17th-century French mathematician Guillaume de l’Hôpital. Although the rule is often attributed to L’Hôpital, the theorem was first introduced to him in 1694 by the Swiss mathematician Johann Bernoulli.

L’Hôpital’s rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I,

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \; or \pm \infty \text{ and } g^{'}(x) \neq 0 \text{ for all } in\ I \text{ with } x \neq c \text{ and } \frac{f'(x)}{g'(x)} \text{ exists, then}$$

The differentiation of the numerator and denominator often simplifies the quotient or converts it to a limit that can be evaluated directly.

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

.

2 The Form 0/0

Supppose $f(x)$ and $g(x)$ are functions which can be expanded by Taylors Theorem in the neighbourhood of $x = a$, Also let $f(a) = 0$ and $g(a) = 0$ Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{1}$$

as

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5} \\
&= \lim_{x\to 0} \frac{\cos x - 1 + x^2/2}{5x^4} \\
&= \lim_{x\to 0} \frac{-\sin x + x}{20x^3} \\
&= \lim_{x\to 0} \frac{-\cos x + 1}{60x^2} \\
&= \lim_{x\to 0} \frac{\sin x}{120x} \\
&= \lim_{x\to 0} \frac{\cos x}{120} = \frac{1}{120}
\end{aligned}$$

3 The Form $\frac{\infty}{\infty}$

Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{2}$$

as Evaluate

$$\begin{aligned}
&\lim_{x\to 0} \frac{\log x}{cotx} \\
&= \lim_{x\to 0} \frac{1/x}{-cosec^2 x} \\
&= \lim_{x\to 0} \frac{-\sin^2 x}{x} \\
&= \lim_{x\to 0} \frac{-2\sin x \cos x}{1} \\
&= \lim_{x\to 0} \frac{-2\times 0\times 1}{1} = 0
\end{aligned}$$

4 The Form $\infty - \infty$

This form can be easily reduced to the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ Then

$$\begin{aligned}
&\lim_{x\to a}\{f(x) - g(x)\} \\
&= \lim_{x\to a}\left\{\frac{1}{1/f(x)} - \frac{1}{1/g(x)}\right\} \\
&= \lim_{x\to a}\frac{\dfrac{1}{f(x)} - \dfrac{1}{g(x)}}{\dfrac{1}{f(x)g(x)}}
\end{aligned}$$

As

$$\begin{aligned}
&\lim_{x\to\frac{\pi}{2}}\left(\sec x - \frac{1}{1-\sin x}\right) \\
&\lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x} - \frac{1}{1-\sin x}\right) \\
&\lim_{x\to\frac{\pi}{2}}\frac{-\cos x + \sin x}{-\sin x + \sin x^2 - \cos^2 x} = \frac{-0+1}{-1+1-0} = \infty
\end{aligned}$$

5 The Form $0 \times \infty$

This form can be easily reduced to the form $\dfrac{0}{0}$ or to the form $\dfrac{\infty}{\infty}$ Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$
Then $\lim_{x\to a} f(x)g(x)$

$$\begin{aligned}
&\lim_{x\to a}\frac{f(x)}{\dfrac{1}{g(x)}} \\
&\lim_{x\to a}\frac{g(x)}{\dfrac{1}{f(x)}}
\end{aligned}$$

We shall reduce the form $0 \times \infty$ to form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ according to our convenience.
As

$$
\begin{aligned}
&\lim_{x\to 0} x \log \sin x \\
&= \lim_{x\to 0} \frac{\log \sin x}{\frac{1}{x}} \\
&= \lim_{x\to 0} \frac{(1/\sin x).\cos x}{-\frac{1}{x^2}} \\
&= \lim_{x\to 0} \frac{-x^2 \cos x}{\sin x} \\
&= \lim_{x\to 0} \frac{x^2 \sin x - 2x \cos x}{\cos x} = 0
\end{aligned}
$$

6 The Forms 1^∞, 0^0, ∞^0

Suppose $\lim_{x\to 0} f(x)^{g(x)}$ takes any one of these three forms.
Then let $y = \lim_{x\to 0} f(x)^{g(x)}$.
Taking logarithm of the both sides, We get $\log y = \lim_{x\to 0} f(x). \log g(x)$
Now in any of the three cases, log y takes the form $0 \times \infty$ which can be evaluated by the process of above section.

$$
\begin{aligned}
&\lim_{x\to 0} (\cos x)^{\cos^2 x} \\
&\log y = \lim_{x\to 0} (\cot^2 x). \log \cos x \\
&= \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x} \\
&= \lim_{x\to 0} \frac{(1/\cos x).(-\cos x)}{2 \tan x \sec^2 x} \\
&= \lim_{x\to 0} \frac{-\tan x}{2 \tan x \sec^2 x} \\
&= \lim_{x\to 0} \frac{1}{-2 \sec^2 x} = -\frac{1}{2}
\end{aligned}
$$

As,

$$
\begin{aligned}
&\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x \\
&\log y = \lim_{x\to\infty} \left\{ x \log(1 + \frac{a}{x}) \right\} \\
&= \lim_{x\to\infty} \frac{\log(1 + \frac{a}{x})}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{a}{1 + \frac{a}{x}} = a
\end{aligned}
$$

Therefore $y = e^a$

7 Algebraic Structure

We first define $n - ary$ operation for $n = 1, 2, \ldots$ on a set X. Let X be a non-empty set. A mapping

$$f : X^n \to X$$

is called an **n-ary** operation for $n = 1, 2, 3\ldots$ on the set X. For $n = 1$, such an opeation is called a **unary opertaion** on X. For $n = 2$, such an opeation is called a **binary opertaion** on X.

$$f : X \times X \to X$$

For $n = 3$, such an opeation is called a **Ternary opertaion** on X.

$$f : X \times X \times X \to X$$

7.1 Algebraic Struture

A non-empty set X together with one or more $n - ary$ operation (n=1,2,...) is called an algebraic structure.

7.2 Binary Operations

Note. In this section, we deal abstractly with operations on pairs (thus the term "binary") of elements of a set. You are familiar with this concept in the settings of addition, subtraction, multiplication, and (except for 0) division of numbers. Two numbers, such as 9 and 3, yield through these four operations, the numbers 12, 6, 27, and 3, respectively. Notice that taking the 9 first and the 3 second affects the result for subtraction and division. That is, order matters for these operations.

7.2.1 Definition.

A binary operation $\star$ on a set S is a function mapping $S \times S$ into S. For each (ordered pair) $(a, b) \in S \times S$, we denote the element $\star((a, b)) \in S$ as $a \star b$.

Example

The easiest examples of binary operations are addition and multiplication on R. We could also consider these operations on different sets, such as Z, Q, or C.
Note. As we'll see, we don't normally think of subtraction and division as binary operations, but instead we think of them in terms of manipulation of inverse elements with respect to addition and multiplication (respectively).

Example

A more exotic example of a binary operation is matrix multiplication on the set of all 2×2 matrices. Notice that "order matters" (and there is, in general, no such thing as "division" here).

8 Definition: Induced Operation

Let $\star$ be a binary operation on set S and let $H \subset S$. Then H is closed under $\star$ if for all $a, b \in H$, we also have a $\star b \in H$. In this case, the binary operation on H given by restricting $\star$ to H is the **induced operation** of $\star$ on H.

Example

Let $E = \{n \in Z | n \text{is even}\}$ and let $O = \{n \in Z | n \text{is odd}\}$. Then, E is closed under addition (and multiplication). However, O is NOT closed under addition (but is closed under multiplication).

Example

Consider the set of all 2×2 invertible matrices. The set is closed under matrix multiplication (recall $(AB)^{-1} = B^{-1}A^{-1}$), but not closed under matrix addition.

8.1 Defination: Groupoid

if $\star$ is a binary operation on a non-empty set S then the pair $(S, \star)$ is called a **Groupoid**.

Example

(N,+) where N is the set of natural number and + is the operation of addition on N is a groupoid, Note that (N,-) groupoid.

8.2 Defination: Semigroups

A binary operation $\star$ on a nonempty set S is called semigroup if satisfy following property.

- S is closed with respect to $\star$. That is $x, y \in S$ then $x \star y \in S$
- Associative if $(a \star b) \star c$=$a \star (b \star c)$ $\forall$ $a, b, c \in S$

Example:

$(N, +)$, $(N, .)$ and $(R, +)$ are all semigroup.

Example:

The Power set $P(S)$ where S is non-empty set together with the operation $\cup$ of union of two sets is a semigroup.

8.3 Defination: Commutative

A binary operation $\star$ on a set S is **commutative** if $a \star b = b \star a$ for all $a, b \in S$.

Example

Matrix mulitplication on the set of all 2×2 matrices is NOT commutative.

Example

Define $\star$ on Q as $a \star b = ab + 1$. Is $\star$ commutative (prove or find a counterexample)?

8.4 Defination: Associative

A binary operation $\star$ on a set S is associative if $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$.

Example

Define $\star$ on Q as $a \star b = ab + 1$. Is $\star$ associative (prove or find a counterexample)?
Notes We will study several algebraic structures by simply producing the "multiplication table" for the structure. For example, if $S = \{a, b, c\}$ and we have:

$$\begin{array}{lll} a \star a = b & a \star b = c & a \star c = b \\ b \star a = a & b \star b = c & b \star c = b \\ c \star a = c & c \star b = b & c \star c = a \end{array}$$

then we represent this binary operation as:

$\star$	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a

Notice that we read this as (ith entry on left) $\star$ (jth entry on top) = (entry in the ith row and jth column).
Notice a$\star$ b = c and b $\star$ a = a, so $\star$ is not commutative.

Notice.: Binary operation $\star$ is commutative if and only if table entries of it are symmetric with respect to the diagonal running from the upper left to the lower right.

Example

Define $a \star b = a/b$ on $Z^+ = N = \{n \in Z | n >\}$. Then N is not closed under $\star$ since, for example, $1 \star 2 =$ $1/2 \notin$ N.

8.5 Defination: Monoids

A semigroup $(S, \star)$ with an identity element with respect to the operation $\star$ is called monoid.
In other words, an algebraic system $(S, \star)$ is called a monoid if following three conditions are satisfied.

- S is closed with respect to $\star$. That is $x, y \in S$ then $x \star y \in S$
- Associative if $(a \star b) \star c$=$a \star (b \star c)$ $\forall$ $a, b, c \in S$
- Existence of identity element. That is there exists an element $e \in S$ such that $e \star x = x \star e = x$ for any $x \in S$

Example:

Consider the power set $P(S)$ of any non-empty set S together with the operation $\cup$ of union of the two sets. Then $(P(S), \cup)$ is a monoid with empty set ϕ as identity element.

Example:

Let $E = \{2, 4, 6, \ldots\}$ and let $+$ be the operation of addition on E. Then $(E, +)$ is a semigroup but it is not a monoid because E does not contain the identity element for the operation $+$.

8.6 Defination:Commutative Monoids

A semigroup $(S, \star)$ with an identity element with respect to the operation $\star$ is called monoid.
In other words, an algebraic system $(S, \star)$ is called a monoid if following three conditions are satisfied.

- S is closed with respect to $\star$. That is $x, y \in S$ then $x \star y \in S$
- Associative if $(a \star b) \star c{=}a \star (b \star c)\ \forall\ a, b, c \in S$
- Existence of identity element. That is there exists an element $e \in S$ such that $e \star x = x \star e = x$ for any $x \in S$
- Commutative if $a \star b = b \star a\ \forall\ a, b \in S$

Theorem:

The identity element in any monoid is unique.
Proof: Let $(S, \star)$ be any monoid. if possible suppose e and $e^{'}$ are two identity element in $(S, \star)$. Since e is an identity element, therefore

$$e \star e^{'} = e^{'} \star e = e^{'} \tag{3}$$

Again, since $e^{'}$ is an identity element, therefore

$$e^{'} \star e = e \star e^{'} = e \tag{4}$$

From (3) and (4) we have

$$e^{'} = e$$

Thus, the identity element in a monoid is unique.

HomeWork

1. $\lim_{x\to\infty} \dfrac{x}{e^x}$
2. $\lim_{x\to\infty} \dfrac{e^x}{x^3}$
3. $\lim_{x\to 1} \dfrac{\log(1-x)}{\cot \pi x}$
4. $\lim_{x\to\infty} \dfrac{\log x}{a^x},\ a > 1$
5. $\lim_{x\to a} \dfrac{\log(x-a)}{\log(e^x - e^a)}$
6. $\lim_{x\to\infty} \dfrac{(\log x)^3}{x}$
7. $\lim_{x\to\infty} \dfrac{(x\log x)^3}{1+x+x^2}$
8. $\lim_{x\to\infty} x \tan \dfrac{1}{x}$
9. $\lim_{x\to 0} \dfrac{\log \tan 2x}{\log \tan 3x}$
10. $\lim_{x\to 0} x \log x$
11. $\lim_{x\to 0} \sin x \log x$
12. $\lim_{x\to\infty} x(a^{\frac{1}{x}} - 1)$
13. $\lim_{x\to\infty} 2^x \sin \dfrac{a}{2^x}$

Question 14:

On N define $\star$ by letting a $\star$ b = c where c is the smallest integer greater than both a and b. Is this a binary operation on N?

Question 15:

Suppose $\star$ is associative on S. Let

$$H = \{a \in S | a \star x = x \star a \forall x \in S\}$$

. Prove that H is closed under $\star$.